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Poisson structures of Calogero–Moser and Ruijsenaars–Schneider models

Inês Aniceto[1], **Jean Avan**[2] **and Antal Jevicki**[3]

[1] CAMGSD, Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
[2] LPTM, Universite de Cergy-Pontoise (CNRS UMR 8089), Saint-Martin2, 2 avenue Adolphe Chauvin, F-95302 Cergy-Pontoise Cedex, France
[3] Department of Physics, Brown University, Box 1843, Providence, RI 02912, USA

E-mail: ianiceto@math.ist.utl.pt, Jean.Avan@u-cergy.fr and Antal_Jevicki@brown.edu

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Abstract

We examine the Hamiltonian structures of some Calogero–Moser and Ruijsenaars–Schneider N-body integrable models. We propose explicit formulations of the bi-Hamiltonian structures for the discrete models and field-theoretical realizations of these structures. We discuss the relevance of these realizations as collective-field theory for the discrete models.

Mathematics Subject Classification: 37K10, 37K05

1. Introduction

Bi-Hamiltonian structures for N-body dynamical systems can be seen as a dual formulation of integrability, in the sense that they substitute a hierarchy of compatible Poisson structures to a hierarchy of commuting Hamiltonians, to establish Liouville integrability of a given system [1, 2]. Our specific interest for this formulation stems here from the conjecture that, in the case of the N-body Ruijsenaars–Schneider (RS) model [3], its higher Hamiltonian structures may be the relevant framework to describe the dynamics of some magnon-type solutions of string theory [4, 5]. Relevance of higher Poisson structures was demonstrated in the associated sine-Gordon theory in [6].

This leads us to a general questioning of the bi-Hamiltonian structures for related integrable discrete N-body systems and continuous realizations thereof. More specifically: the explicit realization of the bi-Hamiltonian structure for the rational Calogero–Moser (CM) model [1, 2, 7][4] is the basis for our construction, leading us toward our current proposition of a bi-Hamiltonian structure for A_n rational RS models and trigonometric CM models.

[4] We would like to thank one of the referees for pointing out [7].

At this point, we wish to make an important remark: a realization of a bi-Hamiltonian structure was proposed long ago for the relativistic *N*-body Toda model (see e.g. [8, 9]) which is a long-range limit of RS dynamics. However, the difficulty in the full CM and RS cases lies (in technical terms) in the dynamical nature of the *r*-matrix structure which precludes the use of canonical definitions a la Sklyanin (such as discussed in [8]) of the second Hamiltonian structure as a direct 'quadratization' of the first Hamiltonian structure encapsulated in any linear *r*-matrix structure (see also [10, 11] and references therein).

In parallel, we propose a realization of these three bi-Hamiltonian structures in terms of continuous field theories, which can be identified, at least in the two CM cases, with the collective-field continuous limit of the discrete systems. Identification in the RS case is more questionable and shall be accordingly dealt with in a further study; we shall only give some comments about it.

We shall successively describe the results for the rational CM, trigonometric CM and rational RS models. We denote in the discrete case 'bi-Hamiltonian structures' only those pairs of compatible Poisson brackets obeying in addition the hierarchy equation

$$\{h_n, \mathcal{O}\}_1 = \{h_{n-1}, \mathcal{O}\}_2,$$

where h_n are the towers of commuting Hamiltonians and $\mathcal{O}$ is any observable.

The bi-Hamiltonian structure for the discrete rational CM model is described in [1] and further justified in [2] by explicit construction of the corresponding deformation of the canonical 1-form by a Nijenhuis-torsion free tensor. We give here an explicit realization of the first two Poisson structures in terms of a collective field $\alpha(x)$. The first one of these is the already known collective-field formulation of the rational CM [12–14]. The formalism was recognized as being suitable for a useful representation of higher conserved charges and symmetries of the *N*-body system [14, 15]. For the second Poisson bracket, one requires a deformation of the Poisson brackets of $\alpha(x)$ together with a change in the realization of the variables, understood from the change in the phase-space volume element in the collective field formulation, precisely related to the differing Poisson structure.

We then discuss the case of the trigonometric CM model. Based on the identification between the second Poisson structures of the rational CM and first Poisson structures of the trigonometric CM, we propose a second Poisson structure for the trigonometric CM. A consistent formulation in the framework of a continuous field theory is proposed in terms of a collective field $\alpha(x)$. The validity of the hierarchy equation for the corresponding two brackets is conjectured in the discrete case from consistency checks on the continuous realization.

We finally address the case of a rational RS model. The first Poisson structure on discrete observables was derived recently [16]; we propose here a direct formulation from the Lax matrix Poisson structure and its key r, s-matrix formulation. Once again the identification of this Poisson structure with the second Poisson structure of the rational CM model allows us to propose a second Poisson structure for the rational RS model, with the hierarchy property. We then construct a field-theoretical realization of this bi-Hamiltonian structure. Its relevance as a collective field theory for rational RS is, as we have indicated, a delicate issue, essentially postponed until further studies.

All matrix indices throughout this paper are taken to vary between 1 and *N*, with *N* being a given finite integer.

2. Bi-Hamiltonian structure for rational Calogero–Moser

This was derived in [1, 2]. It is expressed directly in terms of observables, respectively, $I_k \equiv \frac{1}{k}\,\mathrm{tr}(L^k)$ and $J_\ell = \mathrm{tr}(L^{\ell-1}Q)$, where L is the Lax matrix and Q is the position matrix:

$$L_{ij} = p_i\delta_{ij} + \frac{g}{(q_i - q_j)}(1-\delta_{ij}), \qquad Q = \mathrm{diag}(q_i). \tag{1}$$

From the first canonical Poisson bracket $\{p_i, q_j\}_1 = \delta_{ij}$, one obtains the first Poisson bracket expression for the invariant variables I_k, J_ℓ:

$$\begin{aligned}
&\{I_k, I_m\}_1 = 0,\\
&\{I_k, J_\ell\}_1 = -(k+\ell-2)I_{k+\ell-2},\\
&\{J_k, J_\ell\}_1 = (\ell-k)J_{k+\ell-2}.
\end{aligned} \tag{2}$$

The second bracket is obtained directly by exploiting the reduction scheme yielding L and Q from the original matrix variables, and the construction of an explicit Nijenhuis-torsion free tensor yielding the second Poisson bracket of $T^*\mathfrak{gl}(n)$. It reads

$$\begin{aligned}
&\{I_k, I_m\}_2 = 0,\\
&\{I_k, J_\ell\}_2 = -(k+\ell-1)I_{k+\ell-1},\\
&\{J_k, J_\ell\}_2 = (\ell-k)J_{k+\ell-1}.
\end{aligned} \tag{3}$$

It is not easy to express $\{,\}_2$ in terms of the p, q variables, although it may be a very useful alternative in view of the extension to the trigonometric CM or rational RS models.

Remark. It is easy to check (directly) that these two compatible Poisson bracket structures are in fact one pair amongst any one chosen in the following set:

$$\begin{aligned}
&\{I_k, I_m\}_a = 0,\\
&\{I_k, J_\ell\}_a = -(k+\ell-2+a)\left(1+\frac{\lambda_a}{k}\right)I_{k+\ell-2+a},\\
&\{J_k, J_\ell\}_a = (\ell-k)J_{k+\ell-2+a},
\end{aligned} \tag{4}$$

where a is any integer in $\mathbb{Z}$ and λ_a is an arbitrary c-number. Indeed one has

Theorem 1. *Any linear combination $\{,\}_a + x\{,\}_{a'}$ with $a \neq a'$, $x \in \mathbb{C}$, yields a skew-symmetric associative Poisson bracket.*

One has here a one-parameter (λ_a) multi-Hamiltonian structure when $a \in \mathbb{Z}$. More general mixed brackets $\{I_k, J_\ell\}_a$ may be derived, but we have not solved the general co-boundary equation associated with it.

It will be important soon to specify the third Hamiltonian structure of the hierarchy starting with $\{,\}_0$ and $\{,\}_1$. It can be directly computed using the explicit recursion operator in [2]. It is unambiguously found to be given by $\{,\}_a$ with $a = 3$ and $\lambda_3 = 0$.

3. Realization of the bi-Hamiltonian structure: collective field theory

The collective field theory describing the $N \to \infty$ continuous limit of the N-site CM model was described in [17]. It is obtained as the result of a phase-space integral, over the continuous version of variables p and q, replacing the discrete traces of polynomials of the Lax matrix L (substituted consistently by $p(x)$) and position matrix Q (substituted by $q(x)$). The dynamical variables $\alpha^{\pm}$ are identified with the end points of the p-integration. Their Poisson bracket structure must be determined by consistency with the original Poisson bracket structure of the discrete traces, precisely I_k and J_ℓ. The phase-space integration, however, implies a subtle redefinition of the observables, when higher Hamiltonian structures are to be represented, since the invariant phase-space volume is accordingly redefined.

The first Poisson structure is described by [14]

$$\begin{aligned} I_k &= \int^\alpha \mathrm{d}p\,\mathrm{d}q\,\frac{p^k}{k} \equiv \int \mathrm{d}x\,\frac{\alpha^{k+1}}{k(k+1)}, \\ J_\ell &= \int^\alpha \mathrm{d}p\,\mathrm{d}q\,q\cdot p^{\ell-1} \equiv \int \mathrm{d}x\,x\,\frac{\alpha^\ell}{\ell}, \end{aligned} \tag{5}$$

with the Poisson bracket structure for α given by the first Poisson structure in KdV:

$$\{\alpha(x),\alpha(y)\}_1 = -\delta'(x-y). \tag{6}$$

It is immediate to check that it yields precisely the Poisson brackets $\{,\}_1$.

To obtain the realization of the second Poisson structure in terms of 'collective' fields, we assume that the collective variables I_k and J_ℓ are obtained by a similar integration, over a modified phase-space volume, taking into account the change in the Poisson brackets of the same densities I_k, J_ℓ in terms of p and q. In particular, we assume that the degree in p of the density yielding, respectively, I_k and J_k again differs by one unit. We are thus led to the following general form for the observables:

$$\begin{aligned} I_k &= \int \mathrm{d}x\,\alpha(x)^{k+a} f(k), \\ J_\ell &= \int \mathrm{d}x\,x\alpha(x)^{\ell+a-1} g(\ell) \end{aligned} \tag{7}$$

and the Poisson structure for α, assumed to be polynomial symmetric in α:

$$\{\alpha(x),\alpha(y)\} = -\alpha(x)^{c/2}\alpha(y)^{c/2}\delta'(x-y). \tag{8}$$

Determination of the numbers $a, c, f(k)$ and $g(\ell)$ follows straightforward from plugging (7) and (8) into (3), yielding the following results up to an overall normalization of all k-indexed observables by a factor λ^{k-1} with arbitrary λ (corresponding to an arbitrary renormalization of α).

The second Poisson structure $\{,\}_2$ is realized by

$$\begin{aligned} I_k &= \int^\alpha p^{-1}\,\mathrm{d}p\,\mathrm{d}q \equiv \int \mathrm{d}x\,\frac{\alpha^k}{k^2}, \\ J_\ell &= \int^\alpha p^{-1}\,\mathrm{d}p\,\mathrm{d}q\,qp^{\ell-1} \equiv \int \mathrm{d}x\,x\,\frac{\alpha^{\ell-1}}{\ell-1}, \end{aligned}$$

with the following Poisson brackets for α:

$$\{\alpha(x),\alpha(y)\}_2 = \alpha(x)\alpha(y)\delta'(x-y).$$

Note that the result for the continuous observables is indeed obtained by a change in the phase-space volume $\mathrm{d}p\,\mathrm{d} \to p^{-1}\,\mathrm{d}p\,\mathrm{d}q$. Accordingly, the canonical discrete variable becomes now $\ln p$, and one consistently finds that it is now $\ln\alpha(x)$ which (in the continuous limit) has a canonical Poisson bracket structure. This Poisson bracket is the third in the KdV hierarchy. It thus seems that the second Poisson bracket of KdV $\{\alpha(x),\alpha(y)\} \sim \alpha(x)^{1/2}\alpha(y)^{1/2}\delta'(x-y)$ does not play a role in the CM framework[5].

Also note that although the second discrete Poisson bracket does realize the hierarchy property and is therefore correctly identified as the second Poisson bracket in the rational CM

[5] Here we are referring to the long wavelength limit of the second Poisson bracket, not the full Poisson bracket.

bi-Hamiltonian hierarchy, the 'second' continuous Poisson bracket is not so, since not only the field bracket but also the definition of the observables has to be changed. If for consistency one computes the Poisson bracket of the same variables in terms of continuous fields, it yields instead

$$\left\{h_n, \int \mathrm{d}x\, x \frac{\alpha^{\ell-1}}{\ell-1}\right\}_2 \equiv \left\{h_{n+2}, \int \mathrm{d}x\, x \frac{\alpha^{\ell-1}}{\ell-1}\right\}_1,$$

exhibiting a shift of 2 in the degree of Hamiltonian, from which one inescapably concludes that the continuous realization of the second discrete Hamiltonian structure for rational CM is in fact a third Hamiltonian structure for the collective field theory. The second Hamiltonian structure of the latter corresponds obviously to the second KdV bracket, and is seemingly (as we have said) not manifest in the discrete CM frame.

4. Trigonometric Calogero–Moser model

An algebra of observables for the discrete CM trigonometric model is written [18] in terms of the coordinate matrix[6] $K = \sum_j \exp(q_j)\, e_{jj}$ and Lax matrix $L = \sum_i p_i e_{ii} + \sum_{i\neq j} g \frac{\cos(q_i-q_j)}{\sin(q_i-q_j)} e_{ij}$ using the first canonical Poisson structure $\{p_i, q_j\} = \delta_{ij}$. The overcomplete set of observables,

$$\{W_{mn} = \mathrm{Tr}\, L^m\, \mathrm{e}^{nQ}, m \geqslant 0, n \geqslant 0\},$$

is easily shown to realize a $W_{1+\infty}$ algebra (albeit in a very degenerate representation due to the existence of algebraic relations between the W_{mn} issuing from their realization as $N \times N$ matrices):

$$\{W_{mn}, W_{pq}\}_1 = (mq - np) W_{m+p-1,n+q} + \text{lower order terms}.$$

In order to define a second Poisson structure, following the previous derivation, we shall use as independent variables not the (p_i, q_j) but a subset of algebraically independent observables from the set $\{W_{mn}\}$ such that the change of variables be bijective (at least from a given Weyl chamber for the position and momenta variables, since the discrete permutation over indices is factored out by the use of invariant traces). Guided by the discussion in [2], we see that either $\{W_{m0}, W_{m1}, m \leqslant N\}$ or $\{W_{0m}, W_{1m}, m \leqslant N\}$ provides such a subset. Using the first subset seems *a priori* natural since it contains the Hamiltonians $W_{l,0}$. However, $\{W_{m1}, W_{p1}\}_1 = (mq - np) W_{m+p-1,2}$ + lower order terms. It is, in principle, possible to re-express $W_{m+p-1,2}$ in terms of $W_{k,1}$ and $W_{l,0}$ since these second index 0 and 1 observables provide an algebraically complete set of new variables. However, this re-expression is expected to be quite cumbersome: in particular it will certainly yield nonlinear expressions, suggesting that a consistent guess of a compatible second Poisson bracket will be difficult to formulate.

The second set, however, closes linearly and explicitly under the first Poisson bracket and it is thus this one which we choose to define the Poisson hierarchy. It is also crucial to note that no lower order term appears in its Poisson brackets. It then turns out by simple inspection that the first Poisson structure for trigonometric CM expressed in terms of variables $W_{0,1;m}$ is isomorphic to the second Poisson structure for rational CM. It thus seems natural to propose as a second Poisson structure for trigonometric CM the third Poisson structure of rational CM. In terms of W variables it easily reads

$$\{W_{\mathrm{im}}, W_{jn}\}_2 = (in - jm) W_{i+j-1,m+n+1}. \tag{9}$$

This characterizes $\{,\}_1$ and $\{,\}_2$ as a pair of compatible Poisson structures for trigonometric CM model. However, in order to further characterize $\{,\}_1$ and $\{,\}_2$ as a bi-Hamiltonian

[6] We should introduce some notation here. The matrix e_{ij} is a matrix which has all elements zero except the element ij.

structure for the trigonometric CM model, we need to prove that it realizes the hierarchy equality for evolution of observables:

$$\{W_{m0}, W_{in}\}_2 \equiv \{W_{m+1,0}, W_{in}\}_1, \quad i = 0, 1, \quad m \leqslant N.$$

This is not easy since it implies that one is able to compute the second Poisson bracket or the variables W_{m0}, once again a difficult task given that they are redundant variables and we do not control the lower order terms. We shall now use the collective field description of the continuous limit to at least establish the consistency of this statement.

5. Realization: continuous trigonometric Calogero–Moser model

It is known that for a particular value of the coupling constant the trigonometric CM model is equivalent, at the continuum level, to a free fermion on a circle [17]. This suggests that the collective field theory for trigonometric CM should again be expressed as a phase-space integral, this time over a circle in the q variable, yielding the realization of the first Poisson structure as

$$W_{0m} = \operatorname{Tr} e^{mQ} \quad \text{becomes} \quad W_{0m} = \textstyle\int dx\, e^{mx}\alpha(x),$$

$$W_{1m} = \operatorname{Tr} e^{mQ} L \quad \text{becomes} \quad W_{1m} = \textstyle\int dx\, e^{mx}\frac{\alpha(x)^2}{2},$$

and generically

$$W_{nm} = \operatorname{Tr} e^{mQ} L^n \quad \text{becomes} \quad W_{nm} = \int dx\, e^{mx}\frac{\alpha(x)^{n+1}}{n+1}$$

with the Poisson bracket $\{\alpha(x), \alpha(y)\}_1 = \delta'(x - y)$.

This set of integrated collective-field densities realizes indeed the leading (linear) order of the Poisson bracket algebra for the discrete W_{mn} generators under the first Poisson bracket. Note that a similar property already held in the rational case, when one extended the Poisson algebra to the redundant discrete generators $\operatorname{Tr} L^m Q^n$, realized in the continuum limit as $\int dx\, x^m \frac{\alpha^{n+1}}{n+1}$.

Realization of the second Poisson structure is, strictly speaking, only available at this stage for the generators W_{0m}, W_{1m}. We assume as a generic form for this realization the following monomial integrals:

$$W_{\mathrm{im}} = \int dx\, e^{(m+a)x}\frac{\alpha(x)^{i+1+b}}{i+1+b}.$$

Indeed this is the only way to guarantee that the separate additivity (up to a constant!) of the indices i and m will be preserved in the formulation of the Poisson algebra. The Poisson structure for the field α is taken to be the most generic symmetric monomial expression in α and e^x

$$\{\alpha(x), \alpha(y)\} = e^{\frac{c}{2}(x+y)}(\alpha(x)\alpha(y))^{d/2}\delta'(x - y).$$

Plugging these ansatz for W_{0m}, W_{1m} into the expected algebraic structure yields a unique answer:

$$a = -1, \quad c = 2, \quad b = 0, \quad d = 0.$$

In particular one remarks that it is the new $\tilde{\alpha}(x) \equiv e^{-x}\alpha(x)$ which now realizes a canonical Poisson bracket $\{\tilde{\alpha}(x), \tilde{\alpha}(y)\}_2 = \delta'(x - y)$.

Because this realization is unique, and completely determined by the Poisson brackets of the independent generators W_{0m}, W_{1m}, it seems acceptable to conjecture that it will entail a similar realization for the redundant higher order generators W_{nm}, $n \geqslant 2$. From our previous conjecture, they are represented as

$$W_{nm} = \int \mathrm{d}x\, \mathrm{e}^{(m-1)x} \frac{\alpha(x)^{n+1}}{n+1}.$$

We can now compute at least the leading order of the actual Hamiltonian action on these conjectured continuous observables, implied by the second Poisson structure:

$$\{W_{n0}, W_{\mathrm{im}}\}^{\text{continuous}}_{(2)} = nm\, W_{n+i-1,m+n+1}.$$

If, as we have conjectured, this representation is indeed the continuous representation of the second Poisson structure on all the observables of the trigonometric CM model, this equation guarantees that, at the discrete level, we have

$$\{W_{n0}, W_{\mathrm{im}}\}^{\text{discrete}}_{(2)} = nm\, W_{n+i-1,m+n+1} = \{W_{n+1,0}, W_{\mathrm{im}}\}^{\text{discrete}}_{(1)}$$

up to lower order terms, which are in any case not accessible to the continuous representation. Therefore, it is not inconsistent to characterize $\{,\}^{\text{discrete}}_{(2)}$ as a second Hamiltonian structure in a multi-Hamiltonian hierarchy for the trigonometric CM.

6. Bi-Hamiltonian structure for the rational Ruijsenaars–Schneider model

A consistent construction of a bi-Hamiltonian structure can be formulated on the following lines.

(a) The canonical Poisson structure in terms of the basic variables p and q is again re-expressed as a Poisson structure for the following variables:

$$I_k = \mathrm{Tr}\frac{L^k}{k}, \qquad J_\ell = \mathrm{Tr}\, QL^{\ell-1}, \tag{10}$$

where L is the Lax matrix for rational RS and $Q = \mathrm{diag}\,(q_i)$ as before. Direct derivation of the Poisson structure for these observables now follows from the r-matrix structure of the rational RS Lax matrix L. It is given by

$$L = \sum_{k,j=1}^{N} \frac{\gamma}{q_k - q_j + \gamma} b_j \mathbf{e}_{kj}, \qquad b_k = \mathrm{e}^{p_k} \prod_{j\neq k} \left(1 - \frac{\gamma^2}{(q_k - q_j)^2}\right)^{1/2}. \tag{11}$$

The matrix $\mathbf{e}_{kj}$ is the $N \times N$ matrix with all components being zero except the kj component, which is 1.

The canonical Poisson bracket in the canonical variables q_k, p_j:

$$\{p_k, p_j\}_0 = \{q_k, q_j\}_0 = 0, \qquad \{q_j, p_k\}_0 = \delta_{kj}. \tag{12}$$

becomes, in the q_k, b_j variables:

$$\begin{aligned} &\{q_k, q_j\} = 0, \\ &\{q_k, b_j\} = b_k \delta_{kj}, \\ &\{b_k, b_j\} = \left\{\frac{1}{q_j - q_k + \gamma} - \frac{1}{q_k - q_j + \gamma} + \frac{2(1-\delta_{kj})}{q_k - q_j}\right\} b_k b_j. \end{aligned} \tag{13}$$

This Poisson bracket is quadratic in the Lax matrix [19] L:

$$\{L_1 \overset{\otimes}{,} L_2\} = a_{12} L_1 L_2 - L_1 L_2 d_{12} - L_1 s_{12} L_2 + L_2 s_{21} L_1, \tag{14}$$

where

$$\begin{aligned} d_{12} &= -a_{12}^{\mathrm{CM}} - w, \\ a_{12} &= -a_{12}^{\mathrm{CM}} - s_{12}^{\mathrm{CM}} + s_{21}^{\mathrm{CM}} + w, \\ s_{21} &= s_{12}^{\mathrm{CM}} - w, \\ s_{12} &= s_{21}^{\mathrm{CM}} + w. \end{aligned} \tag{15}$$

The following tensors already existed in the Calogero–Moser case [20][7]:

$$\begin{aligned} a_{12}^{\mathrm{CM}} &= -\sum_{k\neq j} \frac{1}{q_j - q_k} \mathbf{e}_{jk} \otimes \mathbf{e}_{kj}, \\ s_{12}^{\mathrm{CM}} &= \sum_{k\neq j} \frac{1}{q_j - q_k} \mathbf{e}_{jk} \otimes \mathbf{e}_{kk}. \end{aligned} \tag{17}$$

They actually also define [19] the famous non-skew-symmetric dynamical r-matrix of the rational CM model by

$$r_{12}^{\mathrm{CM}} = a_{12}^{\mathrm{CM}} + s_{12}^{\mathrm{CM}}. \tag{18}$$

The tensor w in (15) only appears in the RS model, it is defined by

$$w = \sum_{k\neq j} \frac{1}{q_k - q_j} \mathbf{e}_{kk} \otimes \mathbf{e}_{jj}.$$

Finally, one can see that the tensors in (15) obey the classical consistency relation

$$a_{12} - d_{12} + s_{21} - s_{12} = 0, \tag{19}$$

which we shall see to be necessary for Poisson commutation of the traces.

We also determine the Poisson brackets of the Lax operator with the position operator Q, defined as

$$Q = \sum_k q_k \mathbf{e}_{kk}, \tag{20}$$

obtaining

$$\begin{aligned} \{L_1 \overset{\otimes}{,} Q_2\} &= \sum_{i,j,k} \{L_{ij}, q_k\} \mathbf{e}_{ij} \otimes \mathbf{e}_{kk} = \sum_{i,j,k} \frac{\gamma}{q_i - q_j + \gamma} \{b_j, q_k\} \mathbf{e}_{ij} \otimes \mathbf{e}_{kk} \\ &= -\sum_{i,j} L_{ij} \mathbf{e}_{ij} \otimes \mathbf{e}_{jj} = -L_1^r \cdot \sum_j \mathbf{e}_{jj} \otimes \mathbf{e}_{jj}. \end{aligned} \tag{21}$$

We now re-write the above Poisson brackets using as basic variables the following traces:

$$W_n^m = \mathrm{tr}(L^n Q^m), \qquad m = 0, 1. \tag{22}$$

The simplest of the Poisson brackets is

$$\begin{aligned} \left\{W_n^0, W_m^0\right\} &= \mathrm{tr}_{1,2} \sum_{i,j=1}^{n,m} \{L_1 \overset{\otimes}{,} L_2\} L_1^{n-1} L_2^{m-1} \\ &= mn\, \mathrm{tr}_{1,2}\big((a_{12} - d_{12} - s_{12} + s_{21}) L_1^n L_2^m\big) = 0, \end{aligned}$$

where we used the key consistency relation (19).

[7] For the rational CM model, we have the Lax matrix

$$L^r = \sum_{k=1}^N p_k \,\mathbf{e}_{kk} + \sum_{k\neq j}^N \frac{\gamma}{q_k - q_j} \mathbf{e}_{kj}\,, \tag{16}$$

The next Poisson bracket to be determined is

$$\left\{W_n^0, W_m^1\right\} = \underbrace{\mathrm{tr}_{1,2}\left(\sum_{i,j}\{L_1 \overset{\otimes}{,} L_2\} L_1^{n-1} L_2^{j-1} Q_2 L_2^{m-j}\right)}_{A_{01}} + \underbrace{\mathrm{tr}_{1,2}\left(\sum_{i=1}^{n} L_2^m L_1^{n-i}\{L_1 \overset{\otimes}{,} Q_2\} L_1^{i-1}\right)}_{B_{01}}.$$

The first term is re-written as

$$A_{01} = \mathrm{tr}_{1,2}\left\{n(a_{12} - s_{12}) L_1^n \left[L_2^m, Q_2\right] + \sum_{j=1}^{m} L_2^j Q_2 L_2^{m-j}\left[s_{12}, L_1^n\right]\right\}.$$

We once again made use of the cyclicity of the trace and of relation (19). If we now use the explicit formulas (15), we find that

$$a_{12} - s_{12} = -a_{12}^{\mathrm{CM}} - s_{12}^{\mathrm{CM}} = -r_{12}^{\mathrm{CM}},$$

where the superscript CM corresponds to the Calogero–Moser model. Then we write

$$\mathrm{tr}_{1,2}\left(\sum_{j=1}^{m} L_2^j Q_2 L_2^{m-j}\left[s_{12}, L_1^n\right]\right) = \sum_{j=1}^{m} (L^j Q L^{m-j})_{lk} (s_{12})_{i'j'kl} L_{mn}^n (\delta_{j'm}\delta_{i'n} - \delta_{i'n}\delta_{j'm}) = 0.$$

This allows us to simplify A_{01} even further:

$$A_{01} = \mathrm{tr}_{1,2}\big(n\big(-r_{12}^{\mathrm{CM}}\big) L_1^n \left[L_2^m, Q_2\right]\big) = -n\big(L_{ji}^n (r_{12}^{\mathrm{CM}})_{ijml} L_{lm}^m \big(Q_{mm} - Q_{ll}\big)\big).$$

Using the expression for r_{12}^{CM} in components, we finally find

$$A_{01} = -n \sum_{m \neq l} L_{ml}^n L_{lm}^m = -n\,\mathrm{tr}(L^{m+n}) + n \sum_k L_{kk}^n L_{kk}^m.$$

Let us now turn to the second term of the Poisson brackets B_{01}:

$$B_{01} = \mathrm{tr}_{1,2}\big(n L_2^m \left\{L_1 \overset{\otimes}{,} Q_2\right\} L_1^{n-1}\big) = -n \sum_j L_{jj}^n L_{jj}^m.$$

The final result for this Poisson bracket is just

$$\left\{W_n^0, W_m^1\right\} = -n\,\mathrm{tr}(L^{m+n}) = -n W_{m+n}^0. \tag{23}$$

The final Poisson bracket to determine is

$$\left\{W_n^1, W_m^1\right\} = \underbrace{\mathrm{tr}_{1,2}\left(\sum_{i,j=1}^{n,m}\{L_1 \overset{\otimes}{,} L_2\} L_1^{i-1} Q_1 L_1^{n-i} L_2^{j-1} Q_2 L_2^{m-j}\right)}_{A_{11}}$$
$$+ \underbrace{\mathrm{tr}_{1,2}\left(\sum_{j=1}^{m} L_1^n \{Q_1 \overset{\otimes}{,} L_2\} L_2^{j-1} Q_2 L_2^{m-j}\right)}_{B_{11}} + \underbrace{\mathrm{tr}_{1,2}\left(\sum_{i=1}^{n} L_2^m \{L_1 \overset{\otimes}{,} Q_2\} L_1^{i-1} Q_1 L_1^{n-i}\right)}_{C_{11}}.$$

First of all A_{11} is

$$A_{11} = \mathrm{tr}_{1,2}\left(\sum_{i=1}^{n}(a_{12} - s_{12}) L_1^i Q_1 L_1^{n-i}\left[L_2^m, Q_2\right] + \sum_{j=1}^{m}(a_{12} + s_{21})\left[L_1^n, Q_1\right] L_2^j Q_2 L_2^{m-j}\right)$$
$$+ \mathrm{tr}_{1,2}\big(\big([Q_1, d_{12}]Q_2 + [Q_2, d_{12}]Q_1 + Q_2[Q_1, s_{12}] - Q_1[Q_2, s_{21}]\big) L_1^n L_2^m\big).$$

To simplify this expression, we need a few extra results. The first one is

$$\mathrm{tr}_{1,2}(([Q_1, d_{12}]Q_2 + [Q_2, d_{12}]Q_1) = 0$$

due to the cyclicity of the trace. The second one is

$$\mathrm{tr}_{1,2}\big(Q_2[Q_1, s_{12}] L_1^n L_2^m\big) = (L^m Q)_{ij} (s_{12})_{klji} L_{lk}^n (Q_{kk} - Q_{ll}) = 0,$$

where we have used that $s_{12} = s_{21}^{\mathrm{CM}} + w$ from (15): the w contribution is zero, because this tensor is diagonal on both spaces 1 and 2; the contribution from s_{21}^{CM} is also zero due to this tensor being diagonal in the first space 1. A very similar result can be obtained for

$$\mathrm{tr}_{1,2}\big(Q_1[Q_2, s_{21}]\big)L_1^n L_2^m\big) = 0,$$

but in this case one would need to use $s_{21} = s_{12}^{\mathrm{CM}} - w$ from (15).

With these results, A_{11} boils down to

$$A_{11} = \mathrm{tr}_{1,2}\left(-\sum_{i=1}^{n} r_{12}^{\mathrm{CM}} L_1^i Q_1 L_1^{n-i}\big[L_2^m, Q_2\big] + \sum_{j=1}^{m} r_{21}^{\mathrm{CM}}\big[L_1^n, Q_1\big]L_2^j Q_2 L_2^{m-j}\right).$$

In this last expression, we again used the relations directly derived from (15)

$$a_{12} - s_{12} = -r_{12}^{\mathrm{CM}}, \qquad a_{12} + s_{21} = r_{21}^{\mathrm{CM}}.$$

The two terms in A_{11} are further simplified by the use of r_{12}^{CM} in components

$$-\sum_{i=1}^{n} \mathrm{tr}_{1,2}\big(r_{12}^{\mathrm{CM}} L_1^i Q_1 L_1^{n-i}\big[L_2^m, Q_2\big]\big) = -\sum_{i=1}^{n}\sum_{k\neq l}(L^i Q L^{n-i})_{lk} L_{kl}^m$$

$$= -n\,\mathrm{tr}(QL^{m+n}) + \sum_{i=1}^{n}\sum_{k}(L^i Q L^{n-i})_{kk} L_{kk}^m,$$

and likewise

$$\sum_{j=1}^{m} \mathrm{tr}_{1,2}\big(r_{21}^{\mathrm{CM}}\big[L_1^n, Q_1\big]L_2^j Q_2 L_2^{m-j}\big) = m\,\mathrm{tr}(QL^{m+n}) - \sum_{j=1}^{m}\sum_{k}(L^j Q L^{m-j})_{kk} L_{kk}^n.$$

Finally, A_{11} becomes simply

$$A_{11} = (m-n)\,\mathrm{tr}(QL^{m+n}) + \sum_{i=1}^{n}\sum_{k}(L^i Q L^{n-i})_{kk} L_{kk}^m - \sum_{j=1}^{m}\sum_{k}(L^j Q L^{m-j})_{kk} L_{kk}^n.$$

We still have to determine the other terms B_{11} and C_{11}. Let us proceed with B_{11}:

$$B_{11} = \sum_{j=1}^{m} \mathrm{tr}_{1,2}\left(L_1^n L_2 \cdot \sum_k \mathbf{e}_{kk} \otimes \mathbf{e}_{kk} \cdot L_2^{j-1} Q_2 L_2^{m-j}\right)$$

$$= \sum_{j=1}^{m}\sum_{k} L_{kk}^n (L^j Q L^{m-j})_{kk}.$$

In order to obtain the last line, we have used the fact that

$$\sum_k L_{kk}^n (QL^m)_{kk} - \sum_k L_{kk}^n (L^m Q)_{kk} = \sum_k L_{kk}^n Q_{kk} L_{kk}^m - \sum_k L_{kk}^n L_{kk}^m Q_{kk} = 0.$$

Turning to C_{11} one similarly obtains

$$C_{11} = -\sum_{i=1}^{n}\sum_{k} L_{kk}^m (L^i Q L^{n-i})_{kk}.$$

We finally write the result for the Poisson bracket:

$$\big\{W_n^1, W_m^1\big\} = A_{11} + B_{11} + C_{11} = (m-n)\,\mathrm{tr}(QL^{m+n}) = (m-n)W_{m+n}^1.$$

Summarizing the results obtained for the Poisson brackets of the traces, we have for the rational RS model:

$$\begin{aligned} \left\{W_n^0, W_m^0\right\}_1 &= 0\,, \\ \left\{W_n^0, W_m^1\right\}_1 &= -n W_{m+n}^0\,, \\ \left\{W_n^1, W_m^1\right\}_1 &= (m-n) W_{m+n}^1. \end{aligned} \tag{24}$$

Renormalizing the variables $W_n^{0,1}$ to our variables I_k, J_ℓ, by

$$I_k = \frac{1}{k} W_k^0, \qquad J_\ell = W_{\ell-1}^1,$$

we obtain

$$\begin{aligned} \{I_k, I_\ell\}_1 &= 0, \\ \{J_\ell, I_k\}_1 &= (k+\ell-1) I_{k+\ell-1}, \\ \{J_\ell, J_m\}_1 &= (m-\ell) J_{m+\ell-1}. \end{aligned} \tag{25}$$

Another derivation of these Poisson structure was recently given [16], using the realization of the RS model by KKS reduction [21], thereby bypassing the explicit use of the r-matrix structure.

The key remark here is that this canonical (first) bracket for the rational RS is isomorphic to the second bracket $\{,\}_2$ (with $\lambda_2 = 0$) for the rational CM. This is consistent with the remark in [2] on the formal equality of the canonical symplectic form on $T^*GL(n, \mathbb{C})$, yielding the first Poisson structure of the trigonometric CM model, with the relevant symplectic form yielding the second bracket for the rational CM model, together with the well-known Ruijsenaars duality between trigonometric CM and rational RS, certainly valid at least when the first Poisson structures are considered in both formulations.

(b) Even though a direct computation of the new symplectic form deformed by a Nijenhuis-torsion free tensor (i.e. the new canonical 1-form) is not available for rational RS (lacking an obvious choice of such Nijenhuis-torsion free tensor), we however prove, in view of the explicit computations of section 2, that the natural second Poisson brackets for the rational RS hierarchy are expressed in terms of the observables I_k, J_ℓ, by the form of the third Poisson brackets for rational CM written there.

Precisely

$$\begin{aligned} \{I_k, J_\ell\}_2 &= 0, \\ \{J_\ell, I_k\}_2 &= (k+\ell) I_{k+\ell}, \\ \{J_\ell, J_m\}_2 &= (m-\ell) J_{m+\ell+1}. \end{aligned} \tag{26}$$

Proof.

(1) $\{,\}_2$ is compatible with $\{,\}_1$ as a Poisson bracket structure for the observables I_k, J_ℓ of RS since the Jacobi identity equations for $\{,\}_2 + x\{,\}_1$ are the same as for $\{,\}_3^{\mathrm{CM}} + x\{,\}_2^{\mathrm{CM}}$.

(2) We have the following relation:

$$\{J_k, I_\ell\}_2 \equiv \frac{\mathrm{d}^{(2)}}{\mathrm{d}t_\ell} J_k = \{J_k, I_{\ell+1}\}_1 = \frac{\mathrm{d}^{(1)}}{\mathrm{d}t^{\ell+1}} J_k$$

which now characterizes $\{,\}_1$, $\{,\}_2$ as a bona fide bi-Hamiltonian structure for the RS hierarchy, defined by the set of Hamiltonians $\{I_\ell\}$. □

7. Field-theoretical realization of the Ruijsenaars–Schneider structures

We now propose from first principles a field-theoretical realization of the two (bi-Hamiltonian) Poisson structures previously computed for the rational RS models. The first bracket is realized as

$$I_k = \int \mathrm{d}q \frac{\mathrm{e}^{k\alpha}}{k^2}, \qquad J_\ell = \int \mathrm{d}q\, q \frac{\mathrm{e}^{(\ell-1)\alpha}}{\ell - 1},$$

with the Poisson bracket $\{\alpha(x), \alpha(y)\} = \delta'(x - y)$. The exponential representation in α is motivated by the existence of the Ruijsenaars duality between rational RS and trigonometric CM [22] under exchange of the variables p, q. Accordingly, it appears consistent to assume a dual ($x \leftrightarrow \alpha$) representation in the continuum case for the Poisson structure.

The second Poisson structure is now realized in the continuum, following a similar scheme as in the rational and trigonometric CM case. Assuming that a representation purely in $\mathrm{e}^{\alpha(x)}$ will hold for the p variables, one introduces as an ansatz for the observables the generic form

$$I_k = \int \mathrm{d}q \frac{\mathrm{e}^{(k+a)\alpha}}{k + a}, \qquad J_\ell = \int \mathrm{d}q\, q \frac{\mathrm{e}^{(\ell+a-1)\alpha}}{\ell + a - 1}$$

and similarly for the Poisson bracket

$$\{\alpha(x), \alpha(y)\} = \mathrm{e}^{\frac{c}{2}(\alpha(x)+\alpha(y))}\delta'(x - y).$$

The exponential form for α in the Poisson brackets is required by the exponential form in I_k and J_ℓ, which must be preserved under the Poisson bracket to yield again I and J generators. Plugging these ansatz in the second Poisson bracket structure unambiguously yields $a = -1$ and $c = 2$.

From $c = 2$, it is now seen that $\phi(x) \equiv \mathrm{e}^{-\alpha(x)}$ is a canonical field, $\{\phi(x), \phi(y)\} = \delta'(x - y)$. As in the case of the rational CM model, this field-theoretical realization is better interpreted as a third Poisson bracket for the continuous theory since one obtains again a Hamiltonian evolution with a shift of 2 units in the degree:

$$\{h_n, \mathcal{O}\}^{(2)}_{\text{continuous}} \equiv \{h_{n-2}, \mathcal{O}\}^{(1)}_{\text{continuous}},$$

setting $h_n \equiv \int \mathrm{d}q\, \frac{\mathrm{e}^{n\alpha}}{n}$ in both cases, as consistency requires.

The issue is now whether this field-theoretical realization can be obtained directly as a genuine collective field theory for the RS model. This requires a re-writing of the operators I_k, J_ℓ from a collective field theory perspective, and from that a determination of the Poisson structures that arise, thus confirming our ansatz for these. A collective expression for I_1 is known [23], in terms of quantum MacDonald operators, and one would like to extend the analysis done in [23] to higher conserved quantities I_k and J_ℓ, from their expressions in components found in [22]. Such a generalization, however, is not trivial to obtain, because higher powers of the Lax matrix make such calculations very cumbersome.

Before concluding, let us remark that the case of the trigonometric RS model is much more problematic to deal with at this time, due to the difficulty of defining a Poisson-closed complete subalgebra of observables which could be used as suitable coordinates. In this case, the first Poisson structure of neither the set $\{W_{m0}, W_{m1}, m \leqslant N\}$ nor the dual set $\{W_{0m}, W_{1m}, m \leqslant N\}$ closes linearly. Indeed one has $\left\{W_n^1, W_m^1\right\}_1 = (m - n)W^2_{m+n} + \cdots$ and $\left\{W_1^n, W_1^m\right\}_1 = (m - n)W_2^{n+m} + \cdots$. The difficulty which led us to eliminate the choice of the set $\{W_{m0}, W_{m1}, m \leqslant N\}$ in the trigonometric CM case exists now for both sets.

8. Summary

To conclude, we summarize the results obtained here and the remaining issues regarding the construction of multi-Hamiltonian structures for the N-body models, their realization in continuous field theories and interpretation of those as collective field theories.

A_n *Calogero–Moser, rational*. Bi-Hamiltonian structures were already known in the discrete case [1, 2]. A collective-field realization is proposed, with a consistent 'bi-Hamiltonian' structure and consistently modified phase space.

A_n *Calogero–Moser, trigonometric*. Multiple Poisson structures have been established. Consistent collective-field realizations are proposed, with a consistent 'bi-Hamiltonian' structure. The hierarchy equations for the multiple discrete Poisson structures have not been rigorously established but pass consistency checks.

A_n *Ruijsenaars–Schneider, rational.* A bi-Hamiltonian structure is established in the discrete case. A continuum realization is proposed, with a bi-Hamiltonian structure; identification as collective field theory is yet unproven.

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References

[1] Magri F 1978 A simple model of the integrable Hamiltonian equations *J. Math. Phys.* **19** 1156–62
[2] Bartocci C, Falqui G, Mencattini I, Ortenzi G and Pedroni M 2010 On the geometric origin of the bi-Hamiltonian structure of the Calogero–Moser system *Int. Math. Res. Notices* **2010** 279–96
[3] Ruijsenaars S N M and Schneider H 1986 A new class of integrable systems and its relation to solitons *Ann. Phys.* **170** 370–405
[4] Aniceto I and Jevicki A 2006 Notes on collective field theory of matrix and spin Calogero models *J. Phys. A: Math. Gen.* **39** 12765–92
[5] Aniceto I and Jevicki A 2008 N-body dynamics of giant magnons in $R \times S^2$ arXiv:0810.4548 [hep-th]
[6] Mikhailov A 2005 A nonlocal Poisson bracket of the sine-Gordon model arXiv:hep-th/0511069
[7] Borisov A V and Mamaev I S 1999 Poisson structures and Lie algebras in Hamiltonian mechanics *Izhevsk 60×84 1/16 Hdb* 464 pp (in Russian)
[8] Suris Y B 1993 On the bi-Hamiltonian structure of Toda and relativistic Toda lattices *Phys. Lett.* A **180** 419–29
[9] Damianou P A 1994 Multiple Hamiltonian structures for Toda type systems *J. Math. Phys.* **35** 5511–41
[10] Li L C and Parmentier S 1989 Nonlinear Poisson structures and r-matrices *Commun. Math. Phys.* **125** 545–63
[11] Oevel W and Ragnisco O 1989 R-matrices and higher Poisson brackets for integrable systems *Physica* A **161** 181–220
[12] Das S R and Jevicki A 1990 String field theory and physical interpretation of $D = 1$ strings *Mod. Phys. Lett.* A **5** 1639–50
[13] Avan J and Jevicki A 1991 Quantum integrability and exact eigenstates of the collective string field theory *Phys. Lett.* B **272** 17–24
[14] Avan J and Jevicki A 1991 Classical integrability and higher symmetries of collective string field theory *Phys. Lett.* B **266** 35–41
[15] Jevicki A 1993 Developments in 2d string theory arXiv:hep-th/9309115

[16] Ayadi V and Feher L 2010 On the superintegrability of the rational Ruijsenaars–Schneider model *Phys. Lett.* A **374** 1913–6

[17] Jevicki A 1992 Nonperturbative collective field theory *Nucl. Phys.* B **376** 75–98

[18] Avan J and Billey E 1995 Observable algebras for the rational and trigonometric Euler–Calogero–Moser models *Phys. Lett.* A **198** 183–94

[19] Suris Y B 1997 Why are the Ruijsenaars–Schneider and the Calogero–Moser hierarchies governed by the same *r*-matrix? *Phys. Lett.* A **225** 253–62

[20] Avan J and Talon M 1993 Classical R matrix structure for the Calogero model *Phys. Lett.* B **303** 33–7

[21] Feher L and Klimcik C 2009 On the duality between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider models *J. Phys. A: Math. Theor.* **42** 185202

[22] Ruijsenaars S N M 1987 Complete integrability of relativistic Calogero–Moser systems and elliptic function identities *Commun. Math. Phys.* **110** 191

[23] Awata H, Matsuo Y, Odake S and Shiraishi J 1995 Collective field theory, Calogero–Sutherland model and generalized matrix models *Phys. Lett.* B **347** 49–55